Lemma 97. *Let V be a countable dimensional vector space over $\mathbb{C}$. If $T \in \operatorname{End}(V)$, then there exists a scalar $c \in \mathbb{C}$ such that $T - c \cdot Id_V$ is not invertible on V.*

Proof. Suppose $T - c \cdot Id_V$ is invertible for all $c \in \mathbb{C}$. Then $P(T)$ is invertible for all polynomials $P \in \mathbb{C}[x]$, $P \neq 0$. Thus, if $R = P/Q$ is a rational function with $P, Q \in \mathbb{C}[x]$, we can define $R(T)$ by $P(T) \cdot (Q(T))^{-1}$. This rule defines a map $\mathbb{C}(x) \to \operatorname{End}(V)$. If $v \in V$, $v \neq 0$, and $R \in \mathbb{C}(x)$, $R \neq 0$, then $R(T)v \neq 0$. Hence the map

$$\mathbb{C}(x) \to V, \quad R \mapsto R(T)v,$$

is injective. Since $\mathbb{C}(x)$ has uncountable dimension over $\mathbb{C}$ (for example, the rational functions $(x-a)^{-1}$, $a \in \mathbb{C}$, are linearly independent), we get a contradiction. $\square$

Lemma 100 (Dixmier). *Suppose V is a vector space over $\mathbb{C}$ of countable dimension and that $S \subset \mathrm{End}(V)$ acts irre-*

ducibly. If $T \in \mathrm{End}(V)$ commutes with every element of S, then T is a scalar multiple of the identity operator.

Proof. By Lemma 97, there exists a $c \in \mathbb{C}$ such that $T - c \cdot Id_V$ is not invertible on V. Then $\ker(T - c \cdot Id_V)$ and $\mathrm{Im}(T - c \cdot Id_V)$ are preserved by S, hence these spaces are either $\{0\}$ or V. If $\ker(T - c \cdot Id_V) = \{0\}$ and $\mathrm{Im}(T - c \cdot Id_V) = V$, then T is invertible (there is no topology or continuity involved here). Hence $\ker(T - c \cdot Id_V) = V$ or $\mathrm{Im}(T - c \cdot Id_V) = \{0\}$ and in both cases $T = c \cdot Id_V$. $\square$